Relic Radiation from an Evaporating Black Hole

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Abstract

A non–string–theoretic calculation is presented of the microcanonical entropy of relic integer–spin Hawking radiation — at fixed total energy E — from an evanescent, neutral, non–rotating four–dimensional black hole. The only conserved macroscopic quantity is the total energy E which, for a black hole that evaporates completely, is the total energy of the relic radiation. Through a boundary–value approach, in which data for massless, integer–spin perturbations are set on initial and final space–like hypersurfaces, the statistical–mechanics problem becomes, in effect, a one–dimensional problem, with the 'volume' of the system determined by the real part of the time separation at spatial infinity – the variable conjugate to the total energy. We count the number of field configurations on the final space–like hypersurface that have total energy E, assuming that initial perturbations are weak. We find that the density of states resembles the well–known Cardy formula. The Bekenstein–Hawking entropy is recovered if the real part of the asymptotic time separation is of the order of the semi–classical black-hole life–time. We thereby obtain a statistical interpretation of black–hole entropy. Corrections to the microcanonical entropy are computed, and we find agreement with other approaches in terms of a logarithmic correction to the black–hole area law, which is *universal* (independent of black–hole parameters). This result depends crucially upon the discreteness of the energy levels. We dicuss the similarities of our approach with the transition from the black–hole to the fundamental–string régime in the final stages of black–hole evaporation. In addition, we find that the squared coupling, g^2, which regulates the transition from a black hole to a highly–excited string state, and *vice versa*, can be related to the angle, δ, in the complex–time plane, through which we continue analytically the time separation at spatial infinity. Thus, in this scenario, the strong–coupling régime corresponds to a Euclidean black hole, while the physical limit of a Lorentzian space–time (the limit as $\delta \to 0_+$) corresponds to the weak–coupling régime. This resembles the transition of a black hole to a highly–excited string–like state which subsequently decays into massless particles, thereby avoiding the naked singularity.

1. Introduction

Bekenstein [1] was the first to associate an intrinsic entropy with a black hole, such that black-hole entropy measures one's ignorance about the black hole's internal state. The logarithm of the number of distinct internal configurations of the hole, when one studies the microcanonical ensemble, in which the macroscopic external parameters (mass, charge and angular momentum) are fixed, then determines the microcanonical entropy. These internal configurations are related to different possible pre–collapse configurations, which

result in the formation of a stationary black hole with the same external parameters. In this interpretation, the black–hole entropy is the logarithm of the number of distinct ways in which the black hole may be formed by infalling matter.

In relating the dynamical degrees of freedom of a black hole to its entropy, it was suggested that these degrees of freedom refer to states of *all* fields which are located inside the black hole [2]. In particular, modes of fields inside the black hole in the vicinity of the horizon give the dominant contribution to the entropy. Averaging over the states located outside the hole then enables the statistical–mechanical entropy of the black hole to be determined.

The main aim of our approach is to give the black–hole entropy a statistical–mechanical interpretation in terms of a counting of states. We proceed by considering a (quantum or classical) boundary–value problem, in which data for massless, integer–spin, wave–like perturbations are set on initial and final asymptotically–flat space–like hypersurfaces Σ_I and Σ_F , separated by a large proper time T at infinity [3]. (Fermions will be considered in another paper.) In [3], this led to calculations of the quantum amplitude (not just the probability) to go from (say) an initial spherically–symmetric pre–collapse configuration to a prescribed final distribution of radiative fields after the black hole has evaporated completely. In order to make the classical boundary–value problem well–posed, one must rotate the asymptotic proper time T into the lower complex plane; the real–T hyperbolic boundary–value problem for wave–like field perturbations is not well–posed [4].

Below, it will become clear that the real part of the asymptotic proper time T between the initial and final hypersurfaces plays a major part in the computation of the entropy. Apart from the total ADM energy — the only macroscopic quantity appearing which encodes the three–dimensional geometry of the initial and final hypersurfaces — the only other macroscopic quantity is the one-dimensional 'volume' of the system, here effectively given by T . The 'volume', being conjugate to the fixed total energy at infinity, assumes a large range of values. However, we shall not take the thermodynamic or infinite–volume limit. We show that, when the 'volume' T is of the order of the evaporation time of the black hole, and when the given (fixed) total energy, measured after complete evaporation of the black hole, is E , then the logarithm of the number of final radiation microstates is given by the usual expression for the initial entropy of the black hole, which semi–classically has the form $4\pi\, k_B\, M^2\, (m_p)^{-2}$. This gives a statistical–mechanical interpretation for the black–hole entropy.

The final state on the final space–like hypersurface Σ_F is labelled by a set of variables $\{E, b_j\}$. Here, the $\{b_j\}$ denote the massless, integer–spin 'Fourier' amplitudes [see Eq.(1.6) below], j labels the quantum numbers associated with the relic radiation, and E is the fixed total energy of the relic radiation. For simplicity, consider the example of small perturbations $\varepsilon\phi^{(1)}$, about a spherically–symmetric background $\Phi(t, r)$, of a massless real scalar field $\phi = \Phi + \varepsilon\,\phi^{(1)}$, where ε is a small expansion parameter. One expands the perturbation function $\phi^{(1)}$ in the form [3]

$$\phi^{(1)}(t, r, \theta, \phi) \quad = \quad \frac{1}{r} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\Omega)\;\; R_{\ell m}(t, r) \quad . \tag{1.1}$$

The background metric is taken in the form

$$ds^2 \quad = \quad - \; e^{b(t,r)}\, dt^2 \; + \; e^{a(t,r)}\, dr^2 \; + \; r^2 \left(d\theta^2 \, + \, \sin^2\theta \; d\phi^2 \,\right) \quad . \tag{1.2}$$

The perturbed scalar wave equation at late times leads to the $(\ell\,, m)$ mode equation

$$\left(e^{(b-a)/2}\partial_r\right)^2 R_{\ell m} \;-\; \left(\partial_t\right)^2 R_{\ell m} \;-\; \frac{1}{2}\left(\partial_t(a-b)\right)\left(\partial_t R_{\ell m}\right) \;-\; V_\ell(t,r)\, R_{\ell m} \quad = \quad 0\;, \tag{1.3}$$

where

$$V_\ell(t,r) \quad = \quad \frac{e^{b(t,r)}}{r^2}\left(\ell(\ell+1) \; + \; \frac{2m(t,r)}{r}\right) \quad . \tag{1.4}$$

Here, $m(t\,, r)$ is defined by

$$\exp\Big(-\, a(t,r)\Big) \quad = \quad 1 \; - \; \frac{2m(t,r)}{r} \quad . \tag{1.5}$$

Near the final hypersurface Σ_F , one can decompose the radial wave functions $R_{\ell m}(t,r)$ harmonically with respect to time, and the resulting functions $R_{k\ell}(r)$ can be normalised in a convenient way [3]. With this normalisation, when one works near the final hypersurface Σ_F , one can write

$$\phi^{(1)} \quad = \quad \frac{1}{r}\sum_{\ell m}\int_0^\infty dk \;\; b_{k\ell m} \;\; R_{k\ell}(r) \;\; Y_{\ell m}(\Omega) \quad . \tag{1.6}$$

This defines the b_j coefficients for spin $s \;=\; 0$; thus, j denotes the collective indices $k\ell m$. The condition that E is fixed corresponds physically to a non–equilibrium situation, where there is an exchange of energy between the black hole and infinity, which results (after a long time–interval) in the disappearance of the black hole (and possible later re–appearance, etc.). As there are no other conserved quantities apart from the total energy, an isolated system in equilibrium, having total energy E, would sample all its eigenstates at that energy with equal probability. The amplitudes $\{b_j\}$, however, correspond to different final field/particle configurations (microstates) but identical total energy E (macrostate). The coarse–graining arises from the loss of phase information about the final configuration. The microcanonical entropy, therefore, is the logarithm of the number of different microstates that correspond to the same macrostate.

One would also like to understand in more detail what happens in the final stages of black–hole evaporation. For example, it is expected that, within a semi–classical and purely Lorentzian picture, a naked singularity would form just as the black hole finally disappears. Were the black hole to disappear completely, then there would be a breakdown in the unitary evolution of quantum mechanics [5]. That is, pure quantum states would evolve into mixed states when the information about black–hole states located inside the horizon is lost. Here, though, we follow the approach of [3], in which the proper time T at infinity is displaced into the lower complex half–plane, and so any singularity in the classical solution is avoided. This leads us to consider the microcanonical ensemble;

pure quantum states evolve into pure quantum states, and there is no information loss. In another paper, we give in more detail the resulting unitary description of quantum black–hole evolution [6].

In Sec.2, we compute the total energy of the relic Hawking flux crossing the final hypersurface, in the limit of a large total energy or mass. In Sec.3, we calculate the microcanonical density of states as a function of total energy and 'volume', that is, of the asymptotic–time interval T. In Sec.4, we summarise the theory of the black hole/fundamental string transition in the final stages of evaporation, in order to exhibit the similarities with our approach — in a recent paper [7], a cosmological analogue of the black hole/string transition was described. The idea is that massive elementary particles behave like black holes at large coupling. In string theory, there are very massive states, and large numbers of states of a given mass. In this approach, the large degeneracy is regarded as the origin of the Bekenstein–Hawking entropy. There follows a discussion of the thermodynamics in Sec.5. Sec.5.1 is devoted to deriving the semi–classical Bekenstein–Hawking (B–H) entropy formula through constraints on the one–dimensional volume. In Sec.5.2, we compute the corrections, beyond the semi–classical order, to our microcanonical entropy formula, finding agreement with other approaches as to the *universal* negative leading–order correction. Sec.6 puts some of the ideas of Secs.3 and 5 on a firmer basis, in terms of the black hole/string transition theory of Sec.4. Here, we use the probabilistic interpretation of the theory, derived from the quantum amplitudes associated with the boundary data, and make contact with the Euclidean approach to black–hole evaporation.

2. Energy Constraint

The total energy, E, of the radiation fields on the final hypersurface Σ_F , is given by

$$E \;=\; \int_{\Sigma_F} d^3x\;\sqrt{-\gamma}\;\mathcal{H} \quad , \tag{2.1}$$

where $\mathcal{H} = e^{-b}\,T^{(2)}_{\;tt}$, with $T^{(2)}_{\;\mu\nu}$ being the lowest–order perturbed energy–momentum tensor, which is quadratic in the field perturbations. There is no background matter contribution, in view of the final boundary condition that only relic Hawking radiation is present at late times. The gravitational background geometry is described by a metric $\gamma_{\mu\nu}$, which is spherically symmetric, but time–dependent, being of the form given in Eq.(1.2). The background approximation is described more fully in [3]. At late times, the background metric can be accurately described by a Vaidya metric [8]. To simplify the exposition, since, in principle, fields of different spins s may be present, we shall consider here only a massless spin–0 perturbation, writing the perturbative scalar field as $A^{(1)}(x)$; the further inclusion of spin–1 and spin–2 perturbations is straightforward [3]. On integrating by parts, using the linearised massless scalar–field equation and employing the boundary condition of regularity at the origin, $\{r = 0\}$, together with use of the asymptotic behaviour $r^2 A^{(1)}\left(\partial_r A^{(1)}\right) \;\sim\; r^{-1}$ (or faster) at large r , we find, for our perturbative scalar field:

$$\begin{aligned} E \;=\;& \frac{1}{2}\int d\Omega \int_{\Sigma_F} dr\; r^2\; e^{\frac{1}{2}(a-b)}\left(\dot{A}^{(1)\,2} - A^{(1)}\ddot{A}^{(1)}\right) \\ &- \frac{1}{2}\int d\Omega \int_{\Sigma_F} dr\; r^2\; e^{\frac{1}{2}(a-b)}\; A^{(1)}\,\dot{A}^{(1)}\,\frac{1}{2}\left(\dot{a} - \dot{b}\right) \quad . \end{aligned} \tag{2.2}$$

The second term represents the non–linear coupling between a changing massless scalar field and a changing background geometry. A feature of this term is its resemblance to the action integral for $A^{(1)}(x)$, save for the $\frac{1}{2}(\dot{a}-\dot{b})$ term in the integrand, which arises from the time–variation of the background metric. This interaction term is analogous to the phase and area/number interaction discussed in [9], which is there regarded as leading to the Hawking effect. The phase term is represented by the action, while the $\frac{1}{2}(\dot{a}-\dot{b})$ term is effectively the black–hole energy width, which describes the rate of change of the mass $m(t,r)$ inside a radius r at time t in the Vaidya–like region, where $e^{-a} \simeq e^{b} \simeq 1 - 2Gm(t,r)/r$; this term is identified with the area. Since we are here only concerned with physics as measured on the final hypersurface Σ_F at late times T, the width term will be slowly varying and of order $O(m^{-3})$. If one removes it from the integrand and estimates the action term as being $O(m^2)$ (on dimensional grounds), one finds that the interaction term is $O(m^{-1})$. In [9], the interaction term was indeed shown to be $O(m^{-1})$, thus giving a small contribution for large m. Writing ω for a typical mode frequency of the perturbed scalar field [10], one finds that, in the adiabatic approximation $\omega \gg \frac{1}{2}\left|\dot{a}-\dot{b}\right|$, the energy E decomposes approximately into a sum over modes.

Including now also spin–1 and spin–2 particles, with their polarisation states, the total energy of the final radiation field, in the adiabatic approximation, is

$$E \;=\; M c^2 \;=\; \hbar \sum_{s=0,1,2} \sum_{\ell=s}^{\infty} \sum_{P} \sum_{m=-\ell}^{\ell} \frac{(\ell-s)!}{(\ell+s)!} \int_0^{\infty} d\omega \;\omega\; \left|b_{s\omega\ell mP}\right|^2 \quad , \tag{2.3}$$

where M is the initial ADM mass.

The amplitudes $\{b_{s\omega\ell mP}\}$ relate to the relic radiation reaching $\mathcal{I}^+$, and can be expressed in terms of final amplitudes associated with the space–like hypersurface Σ_F [10]. As described in [3], our boundary–value approach leads to a discrete set of final radiation frequencies, given in the adiabatic approximation by $\{\,\omega_n = 2\pi\, n\, \hat{T}^{-1}\; ;\; n = 0, 1, 2, \ldots\}$, where $\hat{T} = 2\left|T\right|$ and where T is the (complex) asymptotic time–interval at spatial infinity, between the initial and final boundary hypersurfaces Σ_I and Σ_F . This gives

$$\begin{aligned} E \;&=\; \hbar \sum_{s=0,1,2} \sum_{\ell=s}^{\infty} \sum_{P} \sum_{m=-\ell}^{\ell} \frac{(\ell-s)!}{(\ell+s)!} \sum_{n=1}^{n_{\max}} \frac{2\pi}{\hat{T}}\; \omega_n\; \left|b_{sn\ell mP}\right|^2 \\ &=\; \frac{2\pi\hbar}{\hat{T}} \sum_{n=1}^{n_{\max}} n\; N_n \quad , \end{aligned} \tag{2.4}$$

where

$$N_n \;=\; \frac{2\pi}{\hat{T}} \sum_{s=0,1,2} \sum_{\ell=s}^{\infty} \sum_{P} \sum_{m=-\ell}^{\ell} \frac{(\ell-s)!}{(\ell+s)!} \left|b_{sn\ell mP}\right|^2 \;\geq\; 0 \tag{2.5}$$

is the occupation number of the mode n, and we have introduced a frequency cut–off through $n_{\max}$. Therefore, the total excitation energy of the gas of particles is given in terms of the total oscillation quantum number N as

$$E \;=\; E_N \;=\; \frac{hN}{\hat{T}} \quad , \tag{2.6}$$

where

$$N \quad = \quad \sum_{n=1}^{n_{\rm max}} n \, N_n \quad . \tag{2.7}$$

Thus, several states have the same total oscillation number. This equation describes combinatorially the splitting of N into a sum of the numbers $n = 1, \ldots, n_{\rm max}$, with each number n appearing N_n times in the sum. Incidentally, Eq.(2.6) also gives the mean one–dimensional density of states

$$\rho(E) \quad = \quad \frac{\hat{T}(E)}{h} \quad = \quad \frac{V}{2\pi\hbar c} \quad , \tag{2.8}$$

where $\hat{T}$ is the period of the phase–space trajectory with fixed energy E and $V = c\,\hat{T}$, defined below in Eq.(3.1).

Naturally, the quantity $n_{\rm max}$ is given by noting that the radiation energy should not exceed the total energy E. That is,

$$\hbar \, \omega_{n_{\rm max}} \quad = \quad E \quad , \tag{2.9}$$

whence $n_{\rm max} = N$. Thus, Eq.(2.7) implies that the ensemble average value of N_n is unity:

$$\frac{1}{N} \sum_{n=1}^{N} n \, N_n \quad = \quad 1 \quad . \tag{2.10}$$

By ergodicity, the ensemble average is equal to the time average for large N. This result follows from Eqs.(2.6,10), on substituting N for $\hat{T}$, and then taking the continuum limit in the summation.

If the relic Hawking–radiation state is independent of the azimuthal quantum number m, then one can write

$$N_n \quad = \quad \frac{2\pi}{\hat{T}} \sum_{s=0,1,2} \; \sum_{P} \; \sum_{\ell=s}^{\infty} \; c_{s\ell} \; \big| b_{sn\ell P} \big|^2 \quad , \tag{2.11}$$

where

$$c_{s\ell} \quad = \quad \big(2\ell + 1 \big) \; \frac{(\ell - s)!}{(\ell + s)!} \quad , \tag{2.12}$$

allowing for the degeneracy in the angular momentum modes. With a view to simplifying the calculation, a further assumption would be to approximate the black hole as a point radiator, which emits only s–waves. S–wave emission is only sensitive to the radial coordinate, and higher angular modes are less significant. In general, a black body of radius R emits modes with angular momenta $\ell \le \ell_{\rm max} = \omega \, R$, where ω is the radiation frequency. This would introduce a cut–off in the ℓ summation.

3. Density of States

As is well known, the canonical–ensemble approach to black–hole evaporation breaks down in Schwarzschild space–time, as the Hawking temperature, T_H , is inversely related to the black–hole mass, M, whence the specific heat $C = \partial M / \partial T_H$ is negative. Another way of seeing this is to note that the density of black–hole states increases roughly at the rate e^{M^2}, for large M, resulting in a canonical partition function which diverges at large M. Therefore, one should instead employ the microcanonical ensemble, which involves a fixed total energy $E \; = \; M\, c^2$; in this case, the total energy is not permitted to fluctuate. The microcanonical entropy does, however, involve corrections to the Bekenstein–Hawking area law, due to quantum space–time fluctuations at fixed horizon area (equivalent to fixed energy), as seen, for example, in the loop quantum gravity approach [11]. In Sec.5.2, we discuss the nature of these entropy corrections.

In this Section, we calculate the microcanonical total density of states $\Omega(E, V)$. For an isolated system, this counts the number of microscopic configurations in a volume

$$V \quad = \quad c\, \hat{T} \quad , \tag{3.1}$$

possessing energies distributed between zero and E. The energy constraint is the only conservation law that we impose. In the quantum theory, when one is counting, different degrees of freedom correspond to distinct states.

Another way of interpreting the microcanonical density of states is as

$$\Omega(E, V) \quad = \quad \sum_{\acute{N}=1}^{\infty} \; \Omega_{\acute{N}}(E, V) \quad , \tag{3.2}$$

where $\Omega_{\acute{N}}(E, V)$, as below, gives the number density of states $\{N_{\acute{N}}\}$ which contain $\acute{N}$ particles. These $\acute{N}$ particles may be labelled by frequencies, ordered such that $\omega_1 \; \leq \; \omega_2 \; \leq \; \ldots \; \leq \; \omega_{\acute{N}}$. In Eq.(3.2), $\Omega(E, V)$ gives the phase–space volume, up to the energy E , of the system with volume V , where $\acute{N}$ denotes the number of degrees of freedom. We are effectively considering a set of oscillators labelled by $\{b_j\}$, whose phases are random; that is, their location in phase space is unknown or their phase–space trajectories are uncorrelated. If the phases were not random, the motion of the system would not be ergodic, and entropy could not be defined.

In order to determine $\Omega(E, V)$, we first define

$$\lambda(E, V) \quad = \quad sum_{\{N_n\}} \; \theta\Big(E \, - \, \hbar \sum_n \, \omega_n \, N_n \Big) \quad , \tag{3.3}$$

where θ is the Heaviside step function and one sums over all states of the fields, labelled by $\{N_{\acute{N}}\}$. The density of states is defined to be

$$\begin{aligned} \Omega(E, V) \quad &= \quad \frac{\partial \lambda}{\partial E} \; \delta E \\ &= \quad \delta E \; \sum_{\{N_n\}} \; \delta\Big(E \, - \, \hbar \sum_n \, \omega_n \, N_n \Big) \quad , \end{aligned} \tag{3.4}$$

namely, the total number of configurations in the energy range $[E\,,\,E\,+\,\delta E]$. The factor δE is some spread in energy or tolerance — independent of E, so as to make $\Omega(E,V)$ dimensionless — that depends on the precision with which the system has been prepared. One apparently natural condition is that $E \gg \delta E$, and that, for any energy fluctuation, $\Delta E \gg \delta E$. For discrete systems, however, δE has a minimum value, although for continuous systems it can be arbitrarily small — see Sec.5.2.

The number density $\Omega_{\acute{N}}(E,V)$ may similarly be represented in the form:

$$\Omega_{\acute{N}}(E,V) \quad = \quad \delta E \sum_{\{N_{N'}\}} \delta\Big(E \,-\, \hbar \sum_{n=1}^{\acute{N}} \,\omega_n\, N_n\Big) \quad . \tag{3.5}$$

This expression (3.5) may also, in the continuum limit, be re–written as:

$$\Omega_{\acute{N}}(E,V) \quad = \quad \frac{\delta E}{\acute{N}\,!} \int_0^E \prod_{n=1}^{\acute{N}} \, dE_n \;\; \rho(E_n) \;\; \delta\Big(E \,-\, \sum_{n=1}^{\acute{N}} \, E_n\Big) \quad , \tag{3.6}$$

where $E_n \;=\; \hbar\,\omega_n\,N_n\,$, and where $\rho(E_n)$ (the single–particle degeneracy) is given by Eq.(2.8). The factor $\prod_{n=1}^{\acute{N}} \, dE_n\,\rho(E_n)$ counts the number of $\acute{N}$–particle states such that the n–th particle has energy E_n. The $1/\big(\acute{N}\,!\big)$ term accounts for identical particles in classical statistics, while the delta–function constraint counts the number of N–particle states with energy $E\,$; for further details, see [12].

Thus, we consider the set of states $\{N_n\}$ as comprising subsets, labelled by $\acute{N}$, such that $\Omega_{\acute{N}}(E,V)$ is the number density of states possessing $\acute{N}$ particles. One can then evaluate $\Omega_{\acute{N}}(E,V)$, giving

$$\begin{aligned} \Omega_{\acute{N}}(E,V) \quad &= \quad \delta E \;\; \frac{E^{\acute{N}-1}}{(\acute{N}-1)!} \; \prod_{n=1}^{\acute{N}} \; \frac{1}{\hbar\,\omega_n} \\ &= \quad \left(\frac{\delta E}{E}\right) \; \frac{1}{\acute{N}\,!\,(\acute{N}-1)!} \; \left(\frac{\hat{T}\,E}{\hbar}\right)^{\acute{N}} \quad . \end{aligned} \tag{3.7}$$

Now, define a (small) dimensionless parameter g by:

$$g^2 \quad = \quad \frac{2\pi\,G\,M}{c^2\,V} \quad = \quad \left(\frac{M}{m_p}\right)^2 \; N^{-1} \quad , \tag{3.8}$$

where the second equality comes from using Eqs.(2.6, 3.1). We then obtain the energy spectrum [13]:

$$E \quad = \quad g\,E_p\,\sqrt{N} \quad , \tag{3.9}$$

where $E_p \;=\; \big(\hbar c^5/G\big)^{\frac{1}{2}}$ is the Planck energy. Notice that, for $E \;\gg\; E_p\,$, the density $\Omega_{\acute{N}}(E,V)$ increases as g^2 decreases, corresponding to the notion that the phase-space

volume — and therefore also the entropy — increases as the observation time $\hat{T}$ exceeds the dynamical time–scale of the collapsing matter, as the black hole forms and loses its hair. In a related calculation in a cosmological context in [14], the aim was to find a gravitational entropy function which increases monotonically as the system becomes more inhomogeneous.

The microcanonical entropy is then given by:

$$S(E,V) \quad = \quad k_B \ \log\left(\sum_{\acute{N}=1}^{\infty} \ \Omega_{\acute{N}}(E,V) \right) \quad . \tag{3.10}$$

One can approximate the right–hand side of Eq.(3.10) as

$$S(E,V) \quad \sim \quad k_B \ \log\Big(\Omega_{\hat{N}}(E,V) \Big) \quad , \tag{3.11}$$

where $\hat{N}$ is the value of $\acute{N}$ which maximises $\log\Big(\Omega_{\acute{N}}(E,V)\Big)$; that is, the most probable value of $\acute{N}$:

$$\left.\frac{\partial \Omega_{\acute{N}}}{\partial \acute{N}}\right|_{\acute{N}} \quad = \quad \hat{N}(E,V) \quad = \quad 0 \quad . \tag{3.12}$$

Using Stirling's formula for large $\acute{N}$, namely, $\acute{N}! \ \sim \ \sqrt{2\pi}\ e^{-\acute{N}} \ \acute{N}^{\acute{N}+\frac{1}{2}}$, and assuming that $\delta E/E$ is independent of $\acute{N}$ (see Sec.5.2), we obtain

$$\hat{N} \quad \simeq \quad \sqrt{\frac{EV}{\hbar c}} \quad \gg \quad 1 \quad . \tag{3.13}$$

Note that $\hat{N} = 1$ corresponds to an elementary particle, as $\hat{T}$ is then the black–hole Compton time. Eq.(3.13) seems to be related to the proposal in [15] for the maximum number of space–time bits in a spatial region with total volume V and energy E. Therefore, the degeneracy is given by

$$\Omega(E,V) \quad \sim \quad \exp\left(2\sqrt{\frac{EV}{\hbar c}} \right) \quad , \tag{3.14}$$

and

$$S(E,V) \quad \sim \quad 2\,k_B \sqrt{\frac{EV}{\hbar c}} \quad = \quad 2\,k_B\,\hat{N} \quad = \quad 2\,k_B\sqrt{N} \quad . \tag{3.15}$$

This is one of the main results of this paper. Note that the entropy is independent of the gravitational constant G.

In arriving at Eq.(3.15), the $\delta E/E$ term was neglected in comparison with the large $\hat{N}$ term. This will be justified in Sec.5.2. Thus, we find the well–known result that, for ideal bosons in a one–dimensional harmonic–oscillator potential, the asymptotic density of states, as $N \to \infty$, is equal to the number of ways of partitioning an integer N into a sum of other integers [16].

Through a little manipulation, the entropy may be expressed in the form

$$S \quad = \quad \frac{2\,k_B}{g}\left(\frac{E}{E_p}\right) \quad . \tag{3.16}$$

Thus, the entropy is linear in the total energy E. The g–dependence illustrates that, as $g \to 0$, the entropy increases, as one expects in the classical limit.

4. Black–Hole/String Correspondence

One could compare the results in Eqs.(3.9,15) with those in [17], which assert that the entropy of a Schwarzschild black hole in string theory is proportional to $\sqrt{N_s}$, where N_s is the level number of a long excited string which collapses into a Schwarzschild black hole at a critical string coupling.

Susskind [18] originally developed the idea that there is a one–to–one correspondence between quantum Schwarzschild black holes and fundamental string states, building upon the ideas first developed in [19]. One immediate similarity is that the degeneracy of states for a given mass increases very rapidly with mass, both for black holes and for fundamental strings. Yet, the leading string entropy term depends linearly upon the mass, whereas the Bekenstein–Hawking entropy is quadratic. The Correspondence Principle [17], however, aims to match these differing behaviours for a particular string coupling g_c. A related (semi–classical) quantum field theory/quantum string black hole duality was investigated in [20].

The one–dimensional string is characterised by its tension $\sim\ (\ell_s)^{-2}$, where ℓ_s is the string scale, and by a (dimensionless) interaction strength g. Varying the coupling g has an interpretation in terms of a varying dilaton background, where the dilaton is given by a scalar field ϕ, through the relation $g^2 = e^{\phi}$. In practice, we treat ϕ as a constant and g as a parameter. Consider a highly–excited string state and a weak string coupling $g \ll 1$, where, in four dimensions, ℓ_s and the Planck scale $\ell_p = \sqrt{G\hbar/c^3}$ are related through

$$g \quad = \quad \frac{\ell_p}{\ell_s} \quad . \tag{4.1}$$

Thus, we initially consider a string scale far above the Planck scale. By increasing the coupling adiabatically, (in effect, then, by increasing the strength of the gravitational interaction), one reaches a point such that the Schwarzschild radius of the string, given in terms of the string mass scale M_s by $2GM_s/c^2$, becomes larger than ℓ_s; at this point, the string makes a (smooth) transition into a black hole. Conversely, however, had one started with a black hole, and then decreased the coupling, one would find that the size of the black hole would eventually fall below the string scale. The correspondence point would be that at which the black hole makes the transition to a string.

The (closed) string mass spectrum has the form [17]

$$M \quad \sim \quad \frac{\hbar\sqrt{N_s}}{c\,\ell_s} \quad = \quad g\,m_p\sqrt{N_s} \quad , \tag{4.2}$$

where $m_p = \hbar/(\ell_p c)$ is the Planck mass, and $N_s \gg 1$ denotes the level number of a long (highly–excited) string. Therefore, for fixed N_s, the string mass increases as N_s increases. The string spectrum involves an infinite tower of massive states, as well as gravitons and massless spin–one particles (for example). For states in free string theory, with a given mass, the most numerous states are the single–string states, since the string entropy depends linearly upon energy. The resemblance of Eq.(4.2) to Eq.(3.9) is evident.

The Bekenstein–Hawking entropy of a four–dimensional Schwarzschild black hole, S_{B-H}, is given in terms of ℓ_p as

$$S_{B--H} \quad = \quad \frac{4\pi\, k_B\, M^2\, (\ell_p)^2\, c^2}{\hbar^2} \quad , \tag{4.3}$$

where M is the black–hole mass. Thus, at the correspondence point where $M = M_s$ (we assume that the mass does not change over the transition), Eq.(4.3) becomes

$$S_{B--H} \quad \sim \quad \frac{4\pi\, k_B\, (\ell_p)^2\, N_s}{(\ell_s)^2} \quad = \quad 4\pi\, k_B\, g^2\, N_s \quad . \tag{4.4}$$

Therefore, at the critical coupling [17],

$$(g_c)^2 \quad \sim \quad (N_s)^{-\frac{1}{2}} \quad , \tag{4.5}$$

the Bekenstein–Hawking entropy becomes the string entropy

$$S_{\text{string}} \quad \sim \quad 4\pi\, k_B \sqrt{N_s} \quad , \tag{4.6}$$

which resembles Eq.(3.15). One sees that the degrees of freedom of an excited free string can match the entropy of a four–dimensional Schwarzschild black hole. It can also be demonstrated that the sub–leading entropy terms, which depend upon the mass, also agree [21]. In this case, the degrees of freedom of a perturbatively interacting string can be identified with Schwarzschild black–hole states.

Instead of fixing g, and letting the black–hole mass M (and therefore N_s) vary as the black hole evaporates, one can, equivalently, keep the entropy fixed, such that $S_{\text{string}} = S_{B-H}$ (equivalent to keeping the state fixed), while varying g adiabatically [22]. Note that, for $g^2 < (N_s)^{-\frac{1}{2}}$, one has $S_{\text{string}} > S_{B-H}$, and conversely, for the case $g^2 > (N_s)^{-\frac{1}{2}}$, one has $S_{\text{string}} < S_{B-H}$. Thus, if there is some critical mass, M_c, such that $S_{\text{string}} > S_{B-H}$, then a black hole can make a transition to a higher–entropy string state. This mass M_c satisfies $GM_c/c^2 \sim \ell_s$, that is,

$$M_c \quad = \quad M_s\, g^{-2} \quad . \tag{4.7}$$

This mass scale is much larger than the Planck mass m_p, in the case that $g \ll 1$. Further, at this mass, the Hawking temperature equals the Hagedorn temperature

$$T_s \quad = \quad (\hbar c)/(k_B\, b\, \ell_s) \quad , \tag{4.8}$$

where b is a constant which depends on the particular model and on the dimension taken for space–time [18,19]. Dimopoulos and Emparan [23] found that a black hole makes a transition to a state of highly–excited and jagged strings (a string ball) at the mass M_c. The production cross sections of black holes and string balls can be matched at M_c. The string ball then loses mass through evaporation, and at a mass just below M_c, it 'puffs up' to a larger random–walk size $\sim \ell_s \left(M/M_s\right)^{\frac{1}{2}}$ (see below [23]). The string ball continues to evaporate with a quasi–thermal spectrum, at the Hagedorn temperature, during which, its size reduces to $\sim \ell_s$. The black hole can be approximated as a point radiator, as its wavelength is of order $\left(\hbar c\right)/\left(k_B\, T_s\right)$, which is larger than the size of the black hole. Since the specific heat of massive string modes is negative, these cannot remain in equilibrium with an infinite heat bath. Therefore, they evaporate into massless modes, so avoiding the possibility of a naked singularity [19].

In [19], the assumption was made that there is an intermediate heavy–string state, which decays at the Hagedorn temperature into massless strings; that is, into elementary particles such as photons. Zalewski [24], however, suggested that the black hole transforms directly into massless string states, using statistical arguments and the property that the evanescent black hole should have zero angular momentum.

In fact, one can also match the total width for string decay (into the dilaton, graviton and massless fields) with that for quantum black holes [25]. The total string width has the form [26]:

$$\Gamma_s \quad = \quad \lambda\, g^2\, M \quad , \tag{4.9}$$

where λ is a dimensionless numerical factor. A formula analogous to Eq.(4.9) encapsulates the decay widths of all the Standard–Model particles, as well as of topological and non–topological solitons, and of cosmic strings [25]. The black–hole decay rate has the form

$$\Gamma_{BH} \quad = \quad \left| \frac{1}{M}\, \frac{dM}{dt} \right| \quad . \tag{4.10}$$

For the low–energy canonical ensemble, we use the Stefan–Boltzmann law

$$\frac{dM}{dt} \quad \sim \quad - \; \frac{4\pi\, \sigma\, \left(R_s\right)^2\, \left(k_B\, T_{BH}\right)^4}{c^4\, \hbar^2} \quad , \tag{4.11}$$

where σ is a dimensionless constant, the Schwarzschild radius is denoted by $R_s = 2GM/c^2$, and $T_{BH} = \left(\hbar c^3\right)/\left(8\pi G M k_B\right)$ denotes the Hawking temperature. Thus,

$$\Gamma_{BH} \quad = \quad \frac{2\, \sigma\, \left(\ell_p\right)^2}{\hbar^2\, c^4}\, \left(k_B\, T_{BH}\right)^3 \quad . \tag{4.12}$$

As $T_{BH} \,\rightarrow\, T_s$ and $R_s \,\rightarrow\, \ell_s$; that is, as the black hole's temperature increases towards the string temperature during evaporation, then

$$\Gamma_{BH} \quad \rightarrow \quad \frac{2\sigma}{b^3}\, g^2\, M_s \quad , \tag{4.13}$$

which resembles Eq.(4.9) in the case that $M \sim M_s$.

A random–walk interpretation of strings at zero coupling also exists [22,27]. From Eq.(4.6), with $n = \sqrt{N_s}$, the string entropy can be represented as an integer n, up to a factor, giving the number of random–walk steps or number of string bits in a polymer string representation [28]. Each step has length ℓ_s and mass M_s . At zero coupling, the free string effectively has a random–walk size $\sqrt{n}\,\ell_s = (N_s)^{\frac{1}{4}}\,\ell_s \gg \ell_s$, and the total length of the string is $L = n\ell_s$.

5. Thermodynamics

Evidently, from Eq.(3.15), $S(E, V)$ is an extensive function of E and V, both of which determine the global properties of the system at infinity. The corresponding thermodynamical quantities, such as temperature θ, pressure P, specific heat C_V and average total particle number $\bar{N}$, can be computed using the standard formulae

$$\left(\frac{\partial S}{\partial E}\right)_V \equiv \frac{1}{\theta}\,, \quad \beta\, P \equiv \left(\frac{\partial S}{\partial V}\right)_E \,, \quad C_V = -\left[\theta^2 \left(\frac{\partial^2 S}{\partial E^2}\right)_V\right]^{-1} \,, \quad \bar{N} = \frac{\beta P V}{k_B} \quad , \tag{5.1}$$

where $\beta = 1/\bigl(k_B\theta\bigr)$, giving

$$\theta = \frac{1}{k_B}\sqrt{\frac{E\hbar c}{V}} = \frac{\hbar c}{k_B V}\sqrt{N}\,, \;\; P = \frac{k_B E}{V} = k_B\,\rho\,, \;\; C_V = S(E,V) > 0\,, \;\; \bar{N} = \hat{N}\,, \tag{5.2}$$

where ρ is the energy density. The second equation in Eq.(5.1) describes an incompressible fluid, for which sound propagates at the speed of light — the Zel'dovich equation of state. We consider both E and V to be large, such that ρ is finite and $E \gg \hbar c/V$. The thermodynamic limit, in which V is regarded as being arbitrarily large, should not be taken, as our approach is based on considering only finite time–intervals at infinity. Hence, the ideal gas approximation holds, in that interactions between particles are being neglected:

$$P\,V = \hat{N}\,k_B\,\theta \quad . \tag{5.3}$$

The number $\hat{N} = E/\bigl(k_B\theta\bigr)$ is also the average number of particles with energy $k_B\theta$ in which the black hole of total energy E decays. The property that $C_V > 0$ implies that a stable thermal equilibrium can be reached by the radiation at any temperature.

One can express the entropy in terms of the temperature θ and the volume or length V, as (see [29]):

$$S = \frac{(k_B)^2}{\pi\hbar c}\,\theta\, V \quad . \tag{5.4}$$

Thus, the entropy is proportional to the length V, and one may interpret the entropy as comprising bits of length V. One obtains the familiar expression for the entropy if one makes the replacement

$$V \to \frac{1}{6}\,\pi^2\, f\, L \quad , \tag{5.5}$$

where L is a length scale and f denotes the number of field species. Thus,

$$S = 2\pi\, k_B \sqrt{\frac{fp}{6}} \quad , \tag{5.6}$$

where $p = \big(EL\big)/\big(hc\big)$ is dimensionless. This is the Cardy formula for central charge f [16]. Eq.(5.6) would have been obtained if, instead of summing over $\{N_n\}$ configurations, we had summed over $\{N_n^i\}$ configurations such that $N_n = \sum_{i=1}^{f} N_n^i$.

When the particle energy $\hbar\,\omega_n$ is much larger than the corresponding temperature θ, then, for $E \gg \hbar\,\omega_n$, one has

$$N \gg n \gg \sqrt{N} \quad , \tag{5.7}$$

and the microcanonical ensemble should be used. Further, in the very–high–energy tail of the spectrum, where quanta have energies comparable with the total energy E of the system, one has

$$n \simeq N \quad . \tag{5.8}$$

On the other hand, for low energies which obey the condition

$$n \ll \sqrt{N} < N \quad , \tag{5.9}$$

the canonical ensemble description of the gas is valid [30].

Another interesting relationship, following from Eq.(5.2), is

$$\frac{V}{2\pi} = \hbar\, c\, \beta \sqrt{N} = g^{-1}\, \ell_p \sqrt{N} \quad , \tag{5.10}$$

which relates the wave period to the inverse temperature. Thus, in this representation, one interprets V — effectively the time–separation at infinity — as a root–mean–square distance, in which case it represents the distance covered by a Brownian random walk with step–size $g^{-1}\, \ell_p$, taking N to be the total number of steps [31]. The analogue of the string length, ℓ_s , in our theory, is provided by $\hat{\ell}_s$, where

$$\hat{\ell}_s = \frac{V}{2\pi}\, N^{-\frac{1}{2}} = \hbar\, c\, \beta \quad , \tag{5.11}$$

which is just the random–walk step size. In addition, with $\hat{\ell}_s = \hbar/\big(\hat{M}_s\, c\big)$, the mass scale $\hat{M}_s$, analogous to the string mass scale M_s , is

$$\hat{M}_s = \frac{M}{\sqrt{N}} = g\, m_p \quad . \tag{5.12}$$

From Eq.(5.2), one has

$$\theta = \frac{\hbar c}{k_B\, \hat{\ell}_s} \quad , \tag{5.13}$$

which is the analogue of the Hagedorn temperature given in Eq.(4.8).

In the light of Eq.(5.11), for the Euclidean quantum gravity approach to black–hole evaporation, the imaginary–time period is given by the inverse temperature of fields on the background space–time [32].

5.1 Bekenstein-Hawking Entropy

Given a four–dimensional, non–rotating, neutral black hole with horizon area A, the Bekenstein–Hawking entropy has the well–known form

$$S_{B--H} \quad = \quad \frac{k_B A}{4\left(\ell_p\right)^2} \quad = \quad 4\pi\, k_B \left(\frac{M}{m_p}\right)^2 \tag{5.14}$$

This is a semi–classical result, valid for $A \gg \left(\ell_p\right)^2$. Let us now equate Eq.(3.15) with Eq.(5.14), which is valid for $M \gg m_p$:

$$S(E,V) \quad = \quad S_{B--H} \quad . \tag{5.15}$$

Matching the four–dimensional entropy with a two–dimensional entropy is just one way of comparing the thermodynamics of any two physical systems. Strong matching might involve equating partition functions which, then, leads to the matching of all thermodynamical quantities [33]. The weak matching condition, Eq.(5.15), which does not necessarily imply the matching of other thermodynamical quantities, implies that

$$\hat{T}(M) \quad = \quad \left(2\pi\right)^3 \frac{G^2 M^3}{\hbar\, c^4} \quad . \tag{5.16}$$

In the semi–classical approximation, the evaporation time–scale, t_0 , of a non–rotating, neutral black hole of initial mass M, is [34]

$$t_0 \quad = \quad \left(3\gamma\right)^{-1} \frac{G^2 M^3}{\hbar\, c^4} \quad , \tag{5.17}$$

where γ is a dimensionless coefficient, which depends on the number of particle species which can be emitted at a given temperature, and on grey–body factors. Thus, $\hat{T}$ is of the order of $\hat{t_0}$.

Another interesting way to arrive at a time–separation at infinity of the order of t_0 , comes through setting, in Eq.(5.10):

$$g^2 \quad = \quad \left(g_c\right)^2 \quad = \quad \frac{1}{2\pi\sqrt{N}} \quad \ll \quad 1 \quad , \tag{5.18}$$

in which case we arrive back at Eq.(5.16). In other words, the B–H entropy equals the 1–dimensional entropy Eq.(3.15), when Eq.(5.18) is satisfied. Therefore, we seem to have enough configurations on Σ_F to reproduce the entropy of black holes. However, if one considered the case in which $2\pi\, g^2 \sqrt{N} \gg 1$, then one would still be in the black–hole

phase, since $\hat{T} < t_0$. When $2\pi g^2 \sqrt{N} < 1$, one is in a post–evaporation phase (perhaps string–like). Eq.(5.18) is essentially the critical coupling condition of Eq.(4.5).

The microcanonical entropy in Eq.(3.15) exceeds the Bekenstein–Hawking value, if $\hat{T}$ exceeds the right–hand side of Eq.(5.16):

$$V \quad > \quad \left(\frac{2\pi GM}{c^2}\right)^3 \left(\ell_p\right)^{-2} \quad . \tag{5.19}$$

However, for $\hat{T}$ given by Eq.(5.16), one has, from Eq.(5.2), $\theta = 4T_H$, where T_H is the Hawking temperature.

We conclude that, if our final space–like hypersurface, Σ_F, were located so that the time separation at spatial infinity between the initial and final hypersurfaces were given by $\frac{1}{2}\hat{T}$ in Eq.(5.16), then the thermal entropy of the remnant radiation would be of the order of the initial black–hole entropy. In that case, therefore, the thermodynamics of the four–dimensional black hole would be encoded in that of a one–dimensional gas of non–interacting bosonic particles. There have been similar approaches that modelled black holes in any number of dimensions, using a one–dimensional gas of massless particles [35].

As an aside, we mention here briefly the connection between our approach and the idea of strings growing or spreading upon reaching the black–hole stretched horizon, which is located at a distance $\sim \ell_s$ from the event horizon [36]. One can re–write Eq.(3.8) in the following suggestive way:

$$\frac{2\pi GM}{c^3} \quad = \quad g^2\,\hat{T} \quad . \tag{5.20}$$

Setting $t_{\rm cross} = 2r_s/c$, which is the time–scale for light crossing a Schwarzschild black hole of radius $r_s = 2GM/c^2$, then

$$t_{\rm cross} \quad = \quad g^2\left(\frac{2\,\hat{T}}{\pi}\right) \quad . \tag{5.21}$$

Due to the redshift effect, the string and all the information it carries appears, from the perspective of a distant observer, to grow as the string passes through the stretched horizon, while, to a freely–infalling observer, it appears to be a Planck–sized object as it crosses the stretched horizon. The string will spread across the entire black hole in a time [36]

$$t_{\rm spread} \quad = \quad \left(\frac{\ell_p}{\ell_s}\right)^2 \frac{G^2M^3}{\hbar\, c^4} \quad = \quad g^2\,\frac{G^2M^3}{\hbar\, c^4} \quad , \tag{5.22}$$

where g is here defined as in Eq.(3.17). The time $t_{\rm spread}$ is short, compared to the black–hole evaporation time t_0, provided that g is small. One can immediately see the similarities between Eqs.(5.21) and (5.22), in the case that $\hat{T}$ is of the order of the black–hole evaporation time. Furthermore, both $t_{\rm cross}$ and $t_{\rm spread}$ seem to encapsulate the notion of information traversing the entire black hole.

5.2 Corrections to the Microcanonical Entropy

Corrections to the B–H area law are important, and related to the evaporation of the black hole. These corrections to the microcanonical entropy, S_{MC}, due to quantum fluctuations at fixed horizon area A, for $A \gg \left(\ell_p\right)^2$, seem to have a *universal* form, that is, a form which is independent of the particular parameters of the black hole [37]:

$$S_{MC} \quad = \quad S_{B-H} \; - \; \frac{3}{2}\,\log\Big(S_{B-H}\Big) \; + \; \text{constant} \; + \; O\Big(\big(S_{B-H}\big)^{-1}\Big) \quad , \tag{5.23}$$

where S_{B--H} is given in Eq.(5.14). Such corrections can also be interpreted as being due to thermal fluctuations about the black hole's equilibrium configuration. The $-\frac{3}{2}\log\big(S_{B-H}\big)$ term, in particular, holds also for all black holes whose microscopic degrees of freedom are described by an underlying conformal field theory, including BTZ and string–theoretic black holes [37,38]. This term also appears when one computes a corrected Cardy formula for the density of states [39] — see also [41].

We now compute the corrections to Eq.(3.15), of the type given in Eq.(5.23), which arise from quantum fields on the black–hole background (as studied above), and from the reduction in the area or mass of the black hole. The B–H law Eq.(3.15) is, of course, only valid for $A \gg \left(\ell_p\right)^2$. One may either convert the $\acute{N}$ sum in Eq.(3.10) into an integral, and then perform a stationary–phase approximation, or one may compute the sum exactly. A straightforward calculation yields

$$S(E,V) \quad \sim \quad 2k_B\sqrt{\frac{EV}{hc}} \; + \; \frac{1}{2}\,k_B\,\log\left(\sqrt{\frac{EV}{hc}}\right) \; + \; k_B\,\log\left(\frac{\delta E}{E}\right) \; + \; \text{constant} \quad , \tag{5.24}$$

where we have neglected higher–order terms. The $+\frac{1}{2}$ coefficient of the logarithm in Eq.(5.24) is in contrast to the results in [42]. It has been suggested that different quantum theories lead to different quantum corrections to the black–hole entropy [37].

One can, however, make further progress with the $\log\big(\delta E/E\big)$ term in Eq.(5.24). Since the energy levels are discrete, δE has a minimum value, which one may take, naturally, to be the spacing between the particle energy levels:

$$\hbar\left(\omega_{n+1} - \omega_n\right) \quad = \quad \frac{h}{\hat{T}} \quad . \tag{5.25}$$

Thus, for $V \gg \big(hc\big)/\big(4E\big)$ and for

$$\delta E \quad = \quad \hbar\,\omega_0 \quad = \quad \frac{hc}{V} \quad , \tag{5.26}$$

(that is, the condition that the energy spread depends solely on the volume), we find, up to unimportant constant terms, that

$$\Omega(E,V) \quad = \quad 2^{-\frac{3}{2}}\left(\frac{EV}{hc}\right)^{-\frac{3}{4}}\exp\left(2\sqrt{\frac{EV}{hc}}\right) \quad , \tag{5.27}$$

which implies that

$$S(E,V) \quad \sim \quad k_B \left(2\sqrt{N} \; - \; \frac{3}{2}\,\log\!\Big(2\sqrt{N} \Big) \right) \; + \; \text{constant} \quad . \tag{5.28}$$

In terms of the leading $-\frac{3}{2}\log(\quad)$ term, this agrees with [37,39], provided that we identify Eq.(3.15) with the black–hole entropy. Clearly, the discreteness of the energy levels is a crucial element in deriving Eq.(5.28).

One might compare Eq.(5.28) with the calculation of the logarithmic corrections to the Cardy formula for the density of states in a two–dimensional conformal field theory [39]. In [39], there is an arbitrary period, which is chosen so that the central charge is *universal* (independent of black–hole area); this again leads to the $-\frac{3}{2}\log\big(S_{B--H}\big)$ term — see also [40]. A similar conclusion holds in the formulation of the present paper, where the spread in energy is fixed in relation to the period at infinity between the initial and final space–like hypersurfaces, Σ_I and Σ_F , as in Eq.(5.26).

6. Probability Distribution

We now discuss further our previous results for quantum amplitudes in black–hole evaporation [3]. We set approximately spherically–symmetric boundary data on an initial space–like hypersurface, Σ_I , and weak, non–spherical boundary data on a final hypersurface, Σ_F , separated from Σ_I by a proper time T, as measured at spatial infinity. The initial data may be regarded as describing a nearly–spherical configuration of gravity and matter, shortly before collapse to a black hole. The weak final data may be regarded as describing a particular choice of very–late–time radiation. The quantum amplitude to go from the prescribed initial data on Σ_I to the prescribed final data on Σ_F is essentially given by $\exp\big(iS_{\rm cl}\big)$, where $S_{\rm cl}$ is the action of a solution of the classical field equations which proceeds from the initial to the final data (if such exists). In particular, this is expected to hold in the case that the theory is invariant under local supersymmetry [3]. In the present case, this semi–classical expression for the quantum amplitude will only be used in the weak–field limit, as above [43]. In fact, the above boundary–value problem, of finding a classical solution to the wave–like (hyperbolic) equations of a field theory, is not well–posed [4]. The natural cure, following Feynman [43], is to rotate the time–interval T, as measured at spatial infinity, slightly into the lower complex half–plane:

$$T \quad = \quad \big|T\big|\, e^{-\,i\delta} \quad , \tag{6.1}$$

where $\delta \; \big(0 \;\le\; \delta \;\ll\; \pi/2\big)$ is a real phase. One finds that $S_{\rm cl}$ has both its real and imaginary parts non–zero. Hence, from the squared modulus of the quantum amplitude, one arrives at a probability distribution for the weak–field radiation, as measured on the final hypersurface Σ_F [44].

Consider a particular configuration of the f oscillators N^i_n (assumed to be independent), at each level n, associated with the boundary data on Σ_F . As above, we write:

$$N \quad = \quad N\big[N^i_n\big] \qquad , \qquad N_n \quad \equiv \quad \sum_{i=1}^{f} N^i_n \quad , \tag{6.2}$$

The probability density has the form (cf. [44]):

$$P\big[\{N_n^i\}\big] \quad = \quad \Big[Z_f\big(\hat g^2\big)\Big]^{-1} \exp\left(-\sum_{n=1}^{n_{\max}}\sum_{i=1}^{f}\gamma_n\,N_n^i\right) \quad , \tag{6.3}$$

where

$$\gamma_n \quad = \quad 2\pi\,\hat g^2\,n \;+\; 2\,\varepsilon^2\,\rho_n \qquad , \qquad \hat g^2 \quad = \quad \frac{\delta}{2} \quad , \tag{6.4}$$

$$\rho_n \quad = \quad \frac{2\pi}{\hat T}\,\sum\nolimits_{k=-\infty}^{\infty}\,\delta\!\left(\omega_n \;-\; \frac{2\pi k}{\hat T}\right) \quad = \quad \sum\nolimits_{k=-\infty}^{\infty}\,e^{ik\omega_n\hat T} \quad , \tag{6.5}$$

and the 'partition function' is

$$Z_f\big(\hat g^2\big) \quad = \quad \sum_{\{N_n^i\}}\exp\left(-\sum_{n=1}^{n_{\max}}\sum_{i=1}^{f}\gamma_n\,N_n^i\right) \quad . \tag{6.6}$$

The parameter ε is the small perturbation–expansion parameter given just before Eq.(1.1). The $2\pi\,\hat g^2\,n$ term in γ_n arises from the contribution to the classical action from the timelike boundary Σ^∞ at spatial infinity, which joins Σ_I to Σ_F [32]. The ε^2 term in γ_n arises from the weak, non–spherical boundary data for integer–spin perturbations on Σ_F [3].

The $\{N_n^i\}$ sum in Eq.(6.6) is over all sequences of independent occupation numbers N_n^i , such that $n \ge 1$, $1 \le i \le f$. Despite the formal similarity of Eq.(6.6) to a thermodynamical partition function, one should view the parameter $2\pi\,\hat g^2$ as being conjugate to N, and not the energy. With Eqs.(6.1) and (6.5) in mind, this is analogous to a number–phase conjugacy in quantum mechanics, although δ is not an oscillator phase. A standard calculation gives

$$Z_f\big(\hat g^2\big) \quad = \quad \prod_{n=1}^{n_{\max}}\Big(1 \;-\; e^{-\gamma_n}\Big)^{-f} \quad , \tag{6.7}$$

and from Eqs.(6.2,3), we obtain the average

$$\bar N_n \quad = \quad \frac{f}{e^{\gamma_n} - 1} \quad . \tag{6.8}$$

A physical interpretation for $\hat g^2$ (and hence for the phase δ) can be obtained by evaluating

$$N \quad = \quad \sum_{n=1}^{n_{\max}} n\,\bar N_n \tag{6.9}$$

and using Eq.(6.8). Assuming that one can convert the sum to an integral (which is the case when $\hat g^2 \ll 1$), we find, from Eq.(6.3), that

$$2\pi\,\hat g^2 \quad \simeq \quad N^{-\frac{1}{2}}\,\sqrt{\frac{\pi^2 f}{6}} \quad . \tag{6.10}$$

Alternatively, let us assume that $\lambda^n \gtrsim \varepsilon^2 \rho_n$, where $\lambda = 2\pi\,\hat{g}^2$. Then,

$$P\big[\{N_n^i\}\big] \quad = \quad \Big[Z_f(\lambda)\Big]^{-1} e^{-\,\lambda N[\{N_n^i\}]} \quad . \tag{6.11}$$

Define

$$\rho(N) \quad = \quad \sum_{\{N_n^i\}} \delta\Big[N \,-\, N\big(\{N_n^i\}\big)\Big] \quad . \tag{6.12}$$

Then,

$$Z_f(\lambda) \quad = \quad \int_0^\infty dN \;\; \rho(N) \;\; e^{-\lambda N} \quad , \tag{6.13}$$

which is just the Laplace transform of $\rho(N)$. Therefore,

$$\hat{P}\big(N\big) \quad = \quad \Big[Z_f(\lambda)\Big]^{-1} e^{-\,\lambda N}\, \rho\big(N\big) \tag{6.14}$$

satisfies

$$\int_0^\infty dN\; \hat{P}\big(N\big) \quad = \quad 1 \quad . \tag{6.15}$$

Let us make the Ansatz

$$\rho\big(N\big) \quad = \quad c_0\, N^{-\frac{1}{2}\beta}\, \exp\Big(\alpha\,\sqrt{N}\Big) \quad , \tag{6.16}$$

where c_0, α and $\beta > 0$ are all constants. Then, the probability density Eq.(6.14) is peaked around $N = N_0$, where

$$\sqrt{N_0} \quad = \quad \frac{1}{4\lambda}\, \Big[\alpha \,\pm\, \sqrt{\alpha^2 - 8\beta\lambda}\Big] \tag{6.17}$$

If we further assume that $\alpha^2 \gg 8\beta\lambda$ and that $N_0 \gg 1$, we obtain

$$\lambda \quad = \quad 2\pi\,\hat{g}^2 \quad \simeq \quad \frac{\alpha}{2\,\sqrt{N_0}} \quad . \tag{6.18}$$

This agrees with Eq.(6.10), provided that $\alpha = 2\pi\sqrt{f/6}$ and $N = N_0$. The inequality $\alpha^2 \gg 8\beta\lambda$ seems reasonable for large f. Therefore,

$$\rho\big(N\big) \quad = \quad c_0\, N^{-\frac{1}{2}\beta}\, \exp\left(2\pi\sqrt{\frac{fN}{6}}\right) \quad . \tag{6.19}$$

The logarithm of Eq.(6.19) agrees with Eq.(5.6) to leading order. Eqs.(6.10,18) are analogous to the critical string coupling condition of Eq.(4.5). This suggests that the parameter $\hat{g}$ behaves like a string coupling parameter.

There are further indications that the rotation angle δ is analogous to the (squared) string coupling. In the squeezed–states approach to black–hole evaporation [45], the squeeze parameter r_j is related to the frequency ω_j and the black–hole mass M through

$$\tanh r_j \quad = \quad \exp\left(-\,\frac{4\pi G M \omega_j}{c^3}\right) \quad . \tag{6.20}$$

Since $< N_j >= \sinh^2(r_j)$, this gives a thermal distribution

$$< N_j > \quad = \quad \frac{1}{e^{\beta_H \omega_j} - 1} \quad , \tag{6.21}$$

at the Hawking temperature $T_H = \left(k_B\, \beta_H\right)^{-1}$. This spectrum is only valid in the phase for which the black hole emits particles at low energy, (which, of course, corresponds to the majority of the lifetime of the black hole). For the final stages of evaporation, the energy spectrum does not have the canonical form, but a micro–canonical distribution [12]. In the approach of [44], the boundary–value problem of [3] for computing quantum amplitudes for wave–like massless perturbations at late times was related to the squeezed–states formalism. It was shown that the limit $\delta \rightarrow 0$ for the rotation angle in Eq.(6.1) corresponds to a highly–squeezed final state for the relic Hawking radiation, with a squeeze parameter r_j given by

$$\tanh r_j \quad = \quad \exp\Big(-\,2\,\omega_j\,|T|\,\sin\delta\Big) \quad . \tag{6.22}$$

Eqs.(6.20) and (6.22) agree when

$$\frac{hN\sin\delta}{Mc^2} \quad = \quad \frac{4\pi G M}{c^3} \quad . \tag{6.23}$$

Here, N is given in Eq.(2.6). For small δ, this gives Eq.(3.9), with g replaced with $\hat{g}$.

In addition, Damour and Ruffini's approach to black–hole evaporation [46] involved complexifying the Schwarzschild mass M, in order to smooth out large oscillations in the outgoing particle wave–function as one approaches the future event horizon. Whether one continues M into the lower or the upper half of the complex plane depends upon whether one wishes to describe positive– or negative–energy states, respectively. In this procedure, quantum states are smooth across the event horizon, and describe a negative–energy component that falls towards the curvature singularity, together with a positive–energy component that can escape to infinity (depending on its energy). If, following this approach, one sets

$$M \quad = \quad \big|M\big|\ \exp\big(-\,i\delta\big) \quad , \tag{6.24}$$

then, for small δ, one has $M = \big|M\big| - \big(i\,\Gamma\big)/2$, where we define

$$\Gamma \quad = \quad 4\,\hat{g}^2\,\big|M\big| \quad . \tag{6.25}$$

This expression bears a resemblance to the expression in Eq.(4.9) for the width or decay rate, taking $\hat{g}$ as in Eq.(6.5).

This analysis suggests that the angle δ, by which the time–interval at spatial infinity, T, is rotated into the complex plane, plays a rôle analogous to that of the string coupling g. Black–hole evaporation can therefore be viewed in the following way: Initially, δ is non–zero, and is given by its value in the Euclidean régime. In the limit $\delta \to 0_+$ (and, hence, for weak coupling $\hat{g} \to 0$), one moves towards the physical, Lorentzian, régime, and the black hole makes a transition to a 'string–like' state, where it subsequently evaporates into massless particles.

7. Conclusion

In this paper, we have expressed the black–hole entropy *via* a counting of massless field configurations which have a fixed total energy on a post-evaporation space–like hypersurface. We have also explored some of the analogies with the string/black–hole correspondence, incorporating some of the notions of black–hole evaporation in a complexified–time representation.

References

[1] J.D. Bekenstein, Phys. Rev. D **7** (1973) 2333.
[2] J.W. York, Phys. Rev. D **28** (1983) 2929.
[3] A.N.St.J. Farley and P.D. D'Eath, Phys. Lett. B **601** (2004) 184 (arXiv gr-qc/0407086); A.N.St.J. Farley and P.D. D'Eath, Ann. Phys. (N.Y.) **321** (2006) 1334; A.N.St.J. Farley and P.D. D'Eath, "Quantum amplitudes in black–hole evaporation. I: Complex approach", (arXiv gr-qc/0510028); A.N.St.J. Farley and P.D. D'Eath, "Quantum amplitudes in black–hole evaporation. II: Spin–0 amplitude", (arXiv gr-qc/0510029).
[4] P. R. Garabedian, *Partial Differential Equations*, (New York: Wiley) (1964).
[5] S.W. Hawking, Phys. Rev. D **14** (1976) 2460.
[6] A.N.St.J. Farley and P.D.D'Eath, in preparation.
[7] S. Kalyana Rama, "A stringy correspondence principle in cosmology", (arXiv hep-th/0603216).
[8] P.C. Vaidya, Proc. Indian Acad. Sci. **A33**, 264 (1951).
[9] G. Gour, Phys. Rev. D **61** (2000) 124007 (arXiv gr-qc/9906116); G. Gour, Phys. Rev. D **61** (2000) 021501 (arXiv gr-qc/9907066).
[10] A.N.St.J. Farley and P.D. D'Eath, Gen. Relativ. Gravit. **38** (2006) 425 (arXiv gr-qc/0510043).
[11] C. Rovelli, Phys. Rev. Lett. **77** (1996) 3288 (arXiv gr-qc/9603063).
[12] B. Harms and Y. Leblanc, Phys. Rev. D **46** (1992) 2334 (arXiv hep-th/9205021).
[13] J.D. Bekenstein, Lett. Nuovo Cim. **11** (1974) 467.
[14] T. Rothman and P. Anninos, Phys. Rev. D **55** (1997) 1948 (arXiv gr-qc/9612063); T. Rothman, Gen. Relativ. Gravit. **32** (2000) 1185 (arXiv gr-qc/9906002).
[15] N. Sasakura, Prog. Theor. Phys. **102** (1999) 169 (arXiv hep-th/9903146).
[16] J.L. Cardy, Nucl. Phys. B **270** (1986) 186.

[17] G.T. Horowitz and J. Polchinski, Phys. Rev. D **57** (1998) 2557 (arXiv hep-th/9707170); G.T. Horowitz and J. Polchinski, Phys. Rev. D **55** (1997) 6189 (arXiv hep-th/9612146); G.T. Horowitz, "Quantum states of black holes", (arXiv gr-qc/9704072).

[18] L. Susskind, "Some speculations about black hole entropy in string theory", (arXiv hep-th/9309145).

[19] M.J. Bowick, L. Smolin and L.C.R. Wijewardhana, Gen. Relativ. Gravit. **19** (1987) 113; M.J. Bowick, L. Smolin and L.C.R. Wijewardhana, Phys. Rev. Lett. **56** (1986) 424.

[20] N.G. Sánchez, Int. J. Mod. Phys. A **19**, 4173 (2004) (arXiv hep-th/0312018); N. Sánchez, "Black hole emission in string theory and the string phase of black holes", (arXiv hep-th/0106221).

[21] S.N. Solodukhin, Phys. Rev. D **57** (1998) 2410 (arXiv hep-th/9701106); R.K. Kaul, Phys. Rev. D **68**, 024026 (2003) (arXiv hep-th/0302170).

[22] T. Damour, Annalen Phys. **11** (2000) 1 [Grav. Cosmol. Suppl. **6** (2000 ANPYA, 9, 267-277, 2000) 63] (arXiv hep-th/9912224); T. Damour and G. Veneziano, Nucl. Phys. B **568** (2000) 93 (arXiv hep-th/9907030).

[23] S. Dimopoulos and R. Emparan, Phys. Lett. B **526** (2002) 393 (arXiv hep-ph/0108060).

[24] K. Zalewski, Phys. Lett. B **207** (1988) 278.

[25] H.J. de Vega and N.G. Sánchez, Phys. Rev. D **67** (2003) 125019.

[26] J. Dai and J. Polchinski, Phys. Lett. B **220** (1989) 387; R.B. Wilkinson, N. Turok and D. Mitchell, Nucl. Phys. B **332**, 131 (1990).

[27] D. Mitchell and N. Turok, Nucl. Phys. B **294** (1987) 1138.

[28] R.R. Khuri, Nucl. Phys. B **588** (2000) 253 (arXiv hep-th/0006063); R.R. Khuri, Nucl. Phys. B **617** (2001) 365 (arXiv hep-th/0107092); R.R. Khuri, Mod. Phys. Lett. A **13** (1998) 1407 (arXiv gr-qc/9803095); R.R. Khuri, Phys. Lett. B **470** (1999) 73 (arXiv hep-th/9910122); O. Bergman and C.B. Thorn, Nucl. Phys. B **502** (1997) 309 (arXiv hep-th/9702068); C.B. Thorn, "Substructure of string", (arXiv hep-th/9607204).

[29] D.V. Fursaev, "Can one understand black hole entropy without knowing much about quantum gravity?", (arXiv gr-qc/0404038).

[30] S. Das, A. Dasgupta and T. Sarkar, Phys. Rev. D **55** (1997) 7693 (arXiv hep-th/9702075).

[31] A. Krause, "Black holes, space–filling chains and random walks", (arXiv hep-th/0312311).

[32] G.W. Gibbons and S.W. Hawking, Phys. Rev. D **15** (1977) 2752.

[33] S. Das and V. Husain, Class. Quantum Grav. **20** (2003) 4387 (arXiv hep-th/0303089).

[34] D.N. Page, Phys. Rev. D **13** (1976) 198.

[35] A. Ghosh, Phys. Lett. B **425** (1998) 269 (arXiv hep-th/9801064).

[36] L. Susskind, Phys. Rev. Lett. **71** (1993) 2367 (arXiv hep-th/9307168); L. Susskind, Phys. Rev. D **49** (1994) 6606 (arXiv hep-th/9308139); L. Susskind and J. Uglum, "Black holes, interactions, and strings", (arXiv hep-th/9410074).

[37] R.K. Kaul and P. Majumdar, Phys. Rev. Lett. **84** (2000) 5255 (arXiv gr-qc/0002040).

[38] S.S. More, "Higher order corrections to black hole entropy", (arXiv gr-qc/0410071).

[39] S. Carlip, Class. Quantum Grav. **17** (2000) 4175 (arXiv gr-qc/0005017).

[40] M.I. Park, "Testing holographic principle from logarithmic and higher order corrections to black hole entropy", (arXiv hep-th/0402173).
[41] S. Kalyana Rama, Phys. Lett. B **566** (2003) 152 (arXiv hep-th/0304152).
[42] J.l. Jing and M.L. Yan, Phys. Rev. D **63** (2001) 024003 (arXiv gr-qc/0005105).
[43] R.P. Feynman and A.R. Hibbs, *Quantum Mechanics and Path Integrals* (New York: McGraw-Hill) (1965).
[44] A.N.St.J. Farley and P.D. D'Eath, Phys. Lett. B **634** (2006) 419. (arXiv gr–qc/0603092); Class. Quantum Grav. **24** (2007) 105.
[45] L.P. Grishchuk and Y.V. Sidorov, Phys. Rev. D **42** (1990) 3413.
[46] T. Damour and R. Ruffini, Phys. Rev. D **14** (1976) 332.